

# DEFORMATION OF $\mathfrak{osp}(2|2)$ -MODULES OF SYMBOLS

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**ABSTRACT.** We classify deformations of  $\mathfrak{osp}(2|2)$ -module structure on the spaces of symbols  $\mathfrak{S}_d^2$  of differential operators acting on the space of weighted densities  $\mathfrak{F}_\lambda^2$ .

**Keywords:** Cohomology, Orthosymplectic superalgebra, deformation, weighted densities.

**MSC(2010):** 17B56, 53D55, 58H15.

## 1. Introduction

The space of weighted densities of weight  $\mu$  on  $\mathbb{R}$  (or  $\mu$ -densities for short), denoted by:

$$\mathcal{F}_\mu = \{f dx^\mu, f \in C^\infty(\mathbb{R})\}, \quad \mu \in \mathbb{R},$$

is the space of sections of the line bundle  $(T^*\mathbb{R})^{\otimes \mu}$ . The Lie algebra  $\text{Vect}(\mathbb{R})$  of vector fields  $X_h = h \frac{d}{dx}$ , where  $h \in C^\infty(\mathbb{R})$ , acts by the *Lie derivative*. Alternatively, this action can be written as follows:

$$(1.1) \quad X_h \cdot (f dx^\mu) := L_{X_h}^\mu (f dx^\mu) = (hf' + \mu h' f) dx^\mu,$$

where  $f'$  and  $h'$  are  $\frac{df}{dx}$  and  $\frac{dh}{dx}$ .

For  $(\lambda, \mu) \in \mathbb{R}^2$  we consider the space  $D_{\lambda, \mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$  of linear differential operators  $A$  from  $\mathcal{F}_\lambda$  to  $\mathcal{F}_\mu$ . The Lie algebra  $\text{Vect}(\mathbb{R})$  acts on the space  $D_{\lambda, \mu}$  by:

$$(1.2) \quad X_h \cdot A = L_{X_h}^\mu \circ A - A \circ L_{X_h}^\lambda.$$

Each module  $D_{\lambda, \mu}$  has a natural filtration by the order of differential operators; the graded module  $\mathcal{S}_{\lambda, \mu} := \text{gr} D_{\lambda, \mu}$  is called the *space of symbols*. The quotient-module  $D_{\lambda, \mu}^k / D_{\lambda, \mu}^{k-1}$  is isomorphic to  $\mathcal{F}_{\lambda - \mu - k}$ , the isomorphism is provided by the principal symbol  $\sigma_{pr}$  defined by

$$A = \sum_{i=0}^k a_i(x) \partial_x^i \mapsto \sigma_{pr}(A) = a_k(x) (dx)^{\mu - \lambda - k}$$

As a  $\text{Vect}(\mathbb{R})$ -module, the space  $\mathcal{S}_{\lambda, \mu}$  depends only on the difference  $d = \mu - \lambda$ , so that  $\mathcal{S}_{\lambda, \mu}$  can be written as  $\mathcal{S}_d$ , and we have

$$\mathcal{S}_d = \bigoplus_{k=0}^{\infty} \mathcal{F}_{d-k}.$$

Deformation problems appear in various areas of mathematics, in particular in algebra, algebraic and analytic geometry, and mathematical physics. Many powerful technics were developed to determine all related deformation obstructions. The deformation theory of Lie algebras is widely studied. Some general questions of the theory were first considered by Richardson-Neijenhuis [14]. Their approach gave a strong relation between a given structure of Lie algebras and adapted cohomological tools. In fact, according to Richardson-Neijenhuis, deformation theory of modules is closely related to the computation of cohomology. In order to make this statement more precise, given a Lie algebra  $\mathfrak{g}$  and a  $\mathfrak{g}$ -module  $V$ , the infinitesimal deformations of the  $\mathfrak{g}$ -module structure on  $V$ , i.e., deformations that are linear in the parameter of deformation are classified by the first cohomology space  $H^1(\mathfrak{g}, \text{End}(V))$ . Of course, not for every infinitesimal deformation there exists a formal deformation containing the latter as an infinitesimal part.

The obstructions or conditions for which an infinitesimal deformation guarantees existence of a formal deformation, are characterized in terms of cup products (also called the Nijenhuis-Richardson products, see [14]) of non-trivial first cohomology classes. These obstructions belong to the second cohomology space  $H^2(\mathfrak{g}, \text{End}(V))$ . This main result have been used by many authors (see [2], [3], [6], [7], [8], [9] and references therein).

Consider the  $\text{Vect}(\mathbb{R})$ -module  $D_d := \text{Hom}_{\text{diff}}(\mathcal{S}_d, \mathcal{S}_d) = \bigoplus_{i,j \geq 0} D_{d-i, d-j}$ . The space  $D_{\lambda, \mu}$  cannot be isomorphic as a  $\text{Vect}(\mathbb{R})$ -module to the corresponding space of symbols, but it is a deformation of this space in the sense of Richardson-Neijenhuis [14].

By restricting ourselves to the Lie algebra  $\mathfrak{sl}(2)$  which is isomorphic to the Lie subalgebra of  $\text{Vect}(\mathbb{R})$  spanned by

$$\{X_1, X_x, X_{x^2}\},$$

we get families of infinite dimensional  $\mathfrak{sl}(2)$ -modules still denoted by  $\mathcal{F}_\lambda$ ,  $D_{\lambda, \mu}$  and  $\mathcal{S}_d$ .

Now, let us consider the superspace  $\mathbb{R}^{1|n}$  endowed with its standard contact structure defined by the 1-form  $\alpha_n$ , and the Lie superalgebra  $\mathcal{K}(n)$  of contact vector fields on  $\mathbb{R}^{1|n}$ . We introduce the  $\mathcal{K}(n)$ -modules  $\mathfrak{F}_\lambda^n$  of  $\lambda$ -densities on  $\mathbb{R}^{1|n}$  and the  $\mathcal{K}(n)$ -modules of linear differential operators,  $\mathfrak{D}_{\lambda, \mu}^n := \text{Hom}_{\text{diff}}(\mathfrak{F}_\lambda^n, \mathfrak{F}_\mu^n)$ , which are super analogues of the spaces  $\mathcal{F}_\lambda$  and  $D_{\lambda, \mu}$ , respectively. The module  $\mathfrak{D}_{\lambda, \mu}^n$  is filtered:

$$\mathfrak{D}_{\lambda, \mu}^{n, 0} \subset \mathfrak{D}_{\lambda, \mu}^{n, \frac{1}{2}} \subset \mathfrak{D}_{\lambda, \mu}^{n, 1} \subset \mathfrak{D}_{\lambda, \mu}^{n, \frac{3}{2}} \subset \dots \subset \mathfrak{D}_{\lambda, \mu}^{n, \ell - \frac{1}{2}} \subset \mathfrak{D}_{\lambda, \mu}^{n, \ell} \dots$$

The corresponding graded module  $\mathfrak{S}_{\lambda, \mu}^n := \text{gr} \mathfrak{D}_{\lambda, \mu}^n$  is isomorphic to

$$\mathfrak{S}_d^n = \bigoplus_{k=0}^{\infty} \mathfrak{F}_{d - \frac{k}{2}}^n, \quad d = \mu - \lambda.$$

We also consider the  $\mathcal{K}(n)$ -module  $\mathfrak{D}_d^n := \text{Hom}_{\text{diff}}(\mathfrak{S}_d^n, \mathfrak{S}_d^n) = \bigoplus_{i,j \geq 0} \mathfrak{D}_{d - \frac{i}{2}, d - \frac{j}{2}}^n$ .

The Lie superalgebra  $\mathfrak{osp}(n|2)$  is the super analogue of  $\mathfrak{sl}(2)$  and it can be realized as a subalgebra of  $\mathcal{K}(n)$ . The spaces  $\mathfrak{F}_\lambda^n$ ,  $\mathfrak{D}_{\mu, \lambda}^n$  and  $\mathfrak{S}_d^n$  are also  $\mathfrak{osp}(n|2)$ -modules.

We are interested to study the formal deformations of the  $\mathfrak{osp}(n|2)$ -modules  $\mathfrak{S}_d^n$ . According to Nijenhuis-Richardson [14], the space  $H^1(\mathfrak{osp}(n|2), \mathfrak{D}_d^n)$  classifies the infinitesimal deformations of the  $\mathfrak{osp}(n|2)$ -module  $\mathfrak{S}_d^n$  and the obstructions to integrability of a given infinitesimal deformation of  $\mathfrak{S}_d^n$  are elements of  $H^2(\mathfrak{osp}(2|n), \mathfrak{D}_d^n)$ . For  $n = 0$  (classical case), the cohomology spaces  $H_{\text{diff}}^1(\mathfrak{sl}(2), D_{\lambda, \mu})$  and  $H_{\text{diff}}^2(\mathfrak{sl}(2), D_{\lambda, \mu})$  were computed by Lecomte [13]. For  $n = 1$ , Basdouri and Ben Ammar computed the cohomology spaces  $H_{\text{diff}}^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu})$  [5] and they studied the formal deformations of the  $\mathfrak{sl}(2)$ -modules  $\mathcal{S}_d$  and the  $\mathfrak{osp}(1|2)$ -modules  $\mathfrak{S}_d^1$  [6]. They exhibited the necessary and sufficient integrability conditions of a given infinitesimal deformation to a formal one and they proved that any formal deformation is equivalent to its infinitesimal part. This work was generalized for  $n \geq 3$  by Abdaoui, Khalfoun and Laraidh [1] since in this case certain cohomological properties of the Lie superalgebras  $\mathfrak{osp}(n|2)$  are similar. So, there seems to be no difference in results obtained in the study of non-trivial deformations of the natural action of this orthosymplectic Lie superalgebra on the direct sum of the superspaces of weighted densities. However, the case  $n = 2$  is exceptional because of an unexpected isomorphism  $\mathcal{K}(n) \simeq \text{Vect}(\mathbb{R}^{1|1})$  (see [12]) which motivate Ben Fraj and Boujelben in [10] to compute the cohomology space  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}^2)$ .

In this paper we are interested in the case  $n = 2$ , we study the formal deformations of the  $\mathfrak{osp}(2|2)$ -modules  $\mathfrak{S}_d^2$  and we give the necessary and sufficient integrability conditions of a given infinitesimal deformation to a formal one.

## 2. DEFINITIONS AND NOTATIONS

Let  $\mathbb{R}^{1|2}$  be the superspace with coordinates  $(x, \theta_1, \theta_2)$  where  $x$  is the even indeterminate,  $\theta_1$  and  $\theta_2$  are odd indeterminates, i.e.,  $\theta_i\theta_j = -\theta_j\theta_i$ .  $\mathbb{R}^{1|2}$  is equipped with the standard contact structure given by the following 1-form:

$$\alpha_2 = dx + \theta_1 d\theta_1 + \theta_2 d\theta_2.$$

Consider  $C^\infty(\mathbb{R}^{1|2})$  through the space of functions of  $\mathbb{R}^{1|2}$ . A  $C^\infty(\mathbb{R}^{1|2})$  function has the form:

$$f(x, \theta_1, \theta_2) = f_0(x) + f_1(x)\theta_1 + f_2(x)\theta_2 + f_{12}(x)\theta_1\theta_2,$$

where  $f_0, f_1, f_2, f_{12} \in C^\infty(\mathbb{R})$ . We denote by  $|f|$  the parity of an homogeneous function  $f$ , that is,  $|f_0(x)| = |f_{12}(x)\theta_1\theta_2| = 0$  and  $|f_1(x)\theta_1| = |f_2(x)\theta_2| = 1$ . Hereafter, the expression  $(-1)^{|f|}$  will be simply written  $(-1)^f$ .

Let  $\text{Vect}(\mathbb{R}^{1|2})$  the superspace of vector fields on  $\mathbb{R}^{1|2}$ :

$$\text{Vect}(\mathbb{R}^{1|2}) = \left\{ h_0\partial_x + h_1\partial_1 + h_2\partial_2 \mid h_i \in C^\infty(\mathbb{R}^{1|2}) \right\},$$

where  $\partial_x = \frac{\partial}{\partial x}$  and  $\partial_i = \frac{\partial}{\partial \theta_i}$ , and consider the Lie superalgebra  $\mathcal{K}(2)$  of contact vector fields on  $\mathbb{R}^{1|2}$ . That is,  $\mathcal{K}(2)$  is a subalgebra of  $\text{Vect}(\mathbb{R}^{1|2})$  preserving the distribution singled out by the 1-form  $\alpha_2$ :

$$\mathcal{K}(2) = \{ X \in \text{Vect}(\mathbb{R}^{1|2}) \mid \text{there exists } f \in C^\infty(\mathbb{R}^{1|2}) \text{ such that } \mathfrak{L}_X(\alpha_2) = f\alpha_2 \},$$

where  $\mathfrak{L}_X$  is the Lie derivative along the vector field  $X$ .

Consider the vector fields  $\bar{\eta}_i = \partial_i - \theta_i\partial_x$ , any contact vector field on  $\mathbb{R}^{1|2}$  can be expressed as

$$X_f = f\partial_x - \frac{1}{2}(-1)^f \sum_{i=1}^2 \bar{\eta}_i(f)\bar{\eta}_i, \quad \text{where } f \in C^\infty(\mathbb{R}^{1|2}).$$

The contact bracket is defined by  $[X_f, X_g] = X_{\{f, g\}}$  where  $\{, \}$  is the Poisson bracket defined by

$$(2.1) \quad \{f, g\} = fg' - f'g - \frac{1}{2}(-1)^f \sum_{i=1}^2 \bar{\eta}_i(f) \cdot \bar{\eta}_i(g).$$

Then the map  $f \mapsto X_f$  is an isomorphism of Lie superalgebra from  $(C^\infty(\mathbb{R}^{1|2}), \{, \})$  to  $(\mathcal{K}(2), [, ])$ . Thus, via this isomorphism, the Lie superalgebra  $\mathcal{K}(2)$  can be identified to the Lie superalgebra  $C^\infty(\mathbb{R}^{1|2})$  endowed with the Poisson bracket.

We define the Lie superalgebra

$$\mathfrak{osp}(2|2) = \langle H, X, Y, A_1, A_2, B_1, B_2, C \rangle.$$

The elements  $H, X, Y$  and  $C$  are even and the elements  $A_i, B_i$  are odd, the bracket is graded antisymmetric, we denote this property by

$$[U, V] = -(-1)^{UV}[V, U].$$

The non zero brackets are:

$$\begin{aligned} [A_i, A_i] &= 2X, & [X, Y] &= 2H, & [H, X] &= X, \\ [A_i, Y] &= -B_i, & [X, B_i] &= A_i, & [H, A_i] &= \frac{1}{2}A_i, \\ [A_i, B_i] &= 2H, & [B_i, B_i] &= -2Y, & [H, B_i] &= -\frac{1}{2}B_i, \\ [A_1, C] &= \frac{1}{2}A_2, & [B_1, C] &= \frac{1}{2}B_2, & [H, Y] &= -Y, \\ [A_2, C] &= -\frac{1}{2}A_1, & [B_2, C] &= -\frac{1}{2}B_1. \end{aligned}$$

It is well known that  $\mathfrak{osp}(2|2)$  can be realized as a subalgebra of  $\mathcal{K}(2)$ :

$$\mathfrak{osp}(2|2) = \text{Span} \left( 1, x, x^2, x\theta_1, x\theta_2, \theta_1, \theta_2, \theta_1\theta_2 \right),$$

Here,

$$(-x, 1, -x^2, 2\theta_i, 2x\theta_i, \theta_1\theta_2) = (H, X, Y, A_i, B_i, C)$$

We easily see that  $\mathfrak{osp}(1|2)$  is isomorphic to a subalgebra of  $\mathfrak{osp}(2|2)$ :

$$\mathfrak{osp}(1|2) \simeq \mathfrak{osp}(1|2)^i = \text{Span}(1, x, x^2, x\theta_i, \theta_i), \quad i = 1, 2.$$

We define the space of  $\lambda$ -densities as

$$(2.2) \quad \mathfrak{F}_\lambda^2 = \left\{ f(x, \theta_1, \theta_2) \alpha_2^\lambda \mid f(x, \theta_1, \theta_2) \in C^\infty(\mathbb{R}^{1|2}) \right\}.$$

As a vector space,  $\mathfrak{F}_\lambda^2$  is isomorphic to  $C^\infty(\mathbb{R}^{1|2})$ , but the Lie derivative of the density  $g\alpha_2^\lambda$  along the vector field  $f := X_f$  in  $\mathcal{K}(2)$  is now:

$$(2.3) \quad \mathfrak{L}_f^\lambda(g\alpha_2^\lambda) = (\mathfrak{L}_f(g) + \lambda f'g)\alpha_2^\lambda.$$

Here, we restrict ourselves to the subalgebra  $\mathfrak{osp}(2|2)$ , thus we obtain a one-parameter family of  $\mathfrak{osp}(2|2)$ -modules on  $C^\infty(\mathbb{R}^{1|2})$  still denoted by  $\mathfrak{F}_\lambda^2$ . As an  $\mathfrak{osp}(1|2)$ -module, we have

$$(2.4) \quad \mathfrak{F}_\lambda^2 \simeq \mathfrak{F}_\lambda^1 \oplus \Pi(\mathfrak{F}_{\lambda+\frac{1}{2}}^1)$$

where  $\Pi$  is the change of parity operator.

### 3. COHOMOLOGY

Let  $\mathfrak{g}$  be a Lie superalgebra acting on a superspace  $V$ . The space of  $n$ -cochains of  $\mathfrak{g}$  with values in  $V$  is the  $\mathfrak{g}$ -module

$$C^n(\mathfrak{g}, V) := \text{Hom}(\Lambda^n(\mathfrak{g}), V).$$

The *coboundary operator*  $\delta_n : C^n(\mathfrak{g}, V) \rightarrow C^{n+1}(\mathfrak{g}, V)$  is a  $\mathfrak{g}$ -map satisfying  $\delta_n \circ \delta_{n-1} = 0$ . The kernel of  $\delta_n$ , denoted  $Z^n(\mathfrak{g}, V)$ , is the space of *n-cocycles*, among them, the elements in the range of  $\delta_{n-1}$  are called *n-coboundaries*. We denote  $B^n(\mathfrak{g}, V)$  the space of *n-coboundaries*. By definition, the  $n^{\text{th}}$  cohomology space is the quotient space

$$H^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V) / B^n(\mathfrak{g}, V).$$

We will only need the formula of  $\delta_n$  (which will be simply denoted  $\delta$ ) in degrees 0 and 1: for  $v \in C^0(\mathfrak{g}, V) = V$ ,  $\delta v(g) := (-1)^{g^v} g \cdot v$  and for  $\omega \in C^1(\mathfrak{g}, V)$ ,

$$\delta\omega(g, h) := (-1)^{g^\omega} g \cdot \omega(h) - (-1)^{h(g+\omega)} h \cdot \omega(g) - \omega([g, h]) \quad \text{for any } g, h \in \mathfrak{g}.$$

For the general expression of  $\delta_n$  see eg [4].

### 4. DEFORMATION THEORY AND COHOMOLOGY

Let  $\rho_0 : \mathfrak{g} \rightarrow \text{End}(V)$  be an action of a Lie superalgebra  $\mathfrak{g}$  on a vector superspace  $V$ . When studying deformations of the  $\mathfrak{g}$ -action  $\rho_0$ , one usually starts with infinitesimal deformations:

$$(4.1) \quad \rho = \rho_0 + t\omega$$

where  $\omega : \mathfrak{g} \rightarrow \text{End}(V)$  is a linear map and  $t$  is a formal parameter with  $|t| = |\omega|$ . From the homomorphism condition

$$(4.2) \quad [\rho(x), \rho(y)] = \rho([x, y])$$

where  $x, y \in \mathfrak{g}$ , we deduce that  $\omega$  is an 1-cocycle. That is, the linear map  $\omega$  satisfies

$$(4.3) \quad (-1)^{x^\omega} [\rho_0(x), \omega(y)] - (-1)^{y(x+\omega)} [\rho_0(y), \omega(x)] - \omega([x, y]) = 0.$$

Moreover, two infinitesimal deformations  $\rho = \rho_0 + t\omega_1$ , and  $\rho = \rho_0 + t\omega_2$  are equivalents if and only if  $\omega_1 - \omega_2$  is coboundary:

$$(4.4) \quad (\omega_1 - \omega_2)(x) = (-1)^{x^A} [\rho_0, A](x) := \delta A(x)$$

where  $A \in \text{End}(V)$ . So, the space  $H^1(\mathfrak{g}, V)$  determines and classifies infinitesimal deformations up to equivalence.

Now, if  $\dim(H^1(\mathfrak{g}, V)) = m$ , then we choose 1-cocycles  $\omega_1, \dots, \omega_m$  representing a basis of  $H^1(\mathfrak{g}, V)$  and we consider the infinitesimal deformation

$$(4.5) \quad \rho = \rho_0 + \sum_{i=1}^m t_i \omega_i$$

where  $t_1, \dots, t_m$  are independent parameters with  $|t_i| = |\omega_i|$ . We try to extend this infinitesimal deformation to a formal one

$$(4.6) \quad \rho = \rho_0 + \sum_{i=1}^m t_i \omega_i + \sum_{i,j} t_i t_j \rho_{ij}^2 + \dots$$

where  $\rho_{ij}^2, \rho_{ijk}^3, \dots$  are linear maps from  $\mathfrak{g}$  to  $\text{End}(V)$  with  $|\rho_{ij}^2| = |t_i t_j|$ ,  $|\rho_{ijk}^3| = |t_i t_j t_k|$  such that

$$(4.7) \quad [\rho(x), \rho(y)] = \rho([x, y]), \quad x, y \in \mathfrak{g}$$

All the obstructions become from this condition and it is well known that they lie in  $H^2(\mathfrak{g}, V)$ . Thus, we will impose extra algebraic relations on the parameters  $t_1, \dots, t_m$ . Let  $\mathcal{R}$  be an ideal in  $\mathbb{C}[[t_1, \dots, t_m]]$  generated by some set of these relations, the quotient

$$(4.8) \quad \mathcal{A} = \mathbb{C}[[t_1, \dots, t_m]]/\mathcal{R}$$

is a supercommutative associative superalgebra with unity.

## 5. COHOMOLOGY AND DEFORMATION OF $\mathfrak{S}_d^2$

We study the formal deformations of the  $\mathfrak{osp}(2|2)$ -module structure on the space of symbols:

$$\mathfrak{S}_d^2 = \bigoplus_{k \geq 0} \mathfrak{S}_{d-\frac{k}{2}}^2$$

The infinitesimal deformations are described by the cohomology space:

$$H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{S}_d^2) = \bigoplus_{i,j \geq 0} H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{d-\frac{i}{2}, d-\frac{j}{2}}^2)$$

Ben Fraj and Boujelben [10] computed the spaces  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}^2)$ , they showed the following result:

**Theorem 5.1.**

$$(5.1) \quad \dim(H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}^2)) = \begin{cases} 2 & \text{if } \lambda = \mu, \\ 3 & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}) \text{ with } k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, basis for these cohomology spaces are given in [10]. Thus,

i) If  $2d \notin \mathbb{N}$ , then

$$H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{S}_d^2) = \bigoplus_{k \geq 0} H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{d-\frac{k}{2}, d-\frac{k}{2}}^2).$$

The space  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{d-\frac{k}{2}, d-\frac{k}{2}}^2)$  is spanned by:

$$\omega_k(f) = f' \quad \text{and} \quad \tilde{\omega}_k(f) = (2d - k)\bar{\eta}_1 \partial_2 f - (-1)^f (\partial_2 f \bar{\eta}_1 + \theta_2 \bar{\eta}_2 \bar{\eta}_1 f \bar{\eta}_2).$$

ii) If  $2d = m \in \mathbb{N}$ , then

$$H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{S}_d^2) = \bigoplus_{k=1}^m H_{\text{diff}}^1\left(\mathfrak{osp}(2|2), \mathfrak{D}_{-\frac{k}{2}, \frac{k}{2}}^2\right) \oplus \bigoplus_{k=-\infty}^m H_{\text{diff}}^1\left(\mathfrak{osp}(2|2), \mathfrak{D}_{\frac{k}{2}, \frac{k}{2}}^2\right).$$

The space  $H_{\text{diff}}^1\left(\mathfrak{osp}(2|2), \mathfrak{D}_{\frac{k}{2}, \frac{k}{2}}\right)$  is spanned by:

$$(5.2) \quad \begin{aligned} \gamma_k(f) &= f' \\ \tilde{\gamma}_k(f) &= \begin{cases} \bar{\eta}_1 \bar{\eta}_2 f & \text{if } k = 0 \\ k \bar{\eta}_1 \partial_2 f - (-1)^f (\partial_2 f \bar{\eta}_1 + \theta_2 \bar{\eta}_2 \bar{\eta}_1 f \bar{\eta}_2) & \text{if } k \neq 0. \end{cases} \end{aligned}$$

The space  $H_{\text{diff}}^1\left(\mathfrak{osp}(2|2), \mathfrak{D}_{-\frac{k}{2}, \frac{k}{2}}^2\right)$  is spanned by:

$$\begin{aligned} \Gamma_k(f) &= f' \bar{\eta}_1 \bar{\eta}_2^{2k-1} \\ \tilde{\Gamma}_k(f) &= k \bar{\eta}_1 (\partial_2 f) \bar{\eta}_1 \bar{\eta}_2^{2k-1} - (-1)^f \left( \partial_2 f \bar{\eta}_2^{2k+1} - \bar{\eta}_1 (\theta_2 \partial_2 f) \bar{\eta}_1^{2k+1} \right) \\ \bar{\Gamma}_k(f) &= (k-1) f'' \bar{\eta}_1 \bar{\eta}_2^{2k-3} + (-1)^f \left( \bar{\eta}_2 f' \bar{\eta}_1^{2k-1} - \bar{\eta}_1 f' \bar{\eta}_2^{2k-1} \right). \end{aligned}$$

In our study, any infinitesimal deformation of  $\mathfrak{osp}(2|2)$ -module on the space  $\mathfrak{S}_d^2$  is of the form:

$$(5.3) \quad \tilde{\mathfrak{L}} = \mathfrak{L} + \mathfrak{L}^1$$

where

$$\mathfrak{L}^1 = \begin{cases} \sum_{k \geq 0} (a_k \omega_k + b_k \tilde{\omega}_k) & \text{if } 2d \notin \mathbb{N} \\ \sum_{k \leq m} (a_k \gamma_k + b_k \tilde{\gamma}_k) + \sum_{k=1}^m (c_k \Gamma_k + d_k \tilde{\Gamma}_k + e_k \bar{\Gamma}_k) & \text{if } 2d = m \in \mathbb{N}. \end{cases}$$

The coefficients  $a_k, b_k, c_k, d_k, e_k$  are independent parameters.

Now, we extend the infinitesimal deformation (5.3) to a formal one:

$$(5.4) \quad \tilde{\mathfrak{L}} = \mathfrak{L} + \mathfrak{L}^1 + \sum_i P_i^2 \mathfrak{L}_i^2 + \sum_i P_i^3 \mathfrak{L}_i^3 + \dots,$$

where the higher order terms  $\mathfrak{L}_i^2, \mathfrak{L}_i^3, \dots$  are linear maps from  $\mathfrak{osp}(2|2)$  to  $\text{End}(\mathfrak{S}_d^2)$  such that the map

$$(5.5) \quad \tilde{\mathfrak{L}} : \mathfrak{osp}(2|2) \rightarrow \mathbb{C}[[a_k, b_k, c_k, d_k, e_k]] \otimes \text{End}(\mathfrak{S}_d^2),$$

satisfies the homomorphism condition

$$(5.6) \quad \tilde{\mathfrak{L}}_{[f,g]} = [\tilde{\mathfrak{L}}_f, \tilde{\mathfrak{L}}_g].$$

$P_i^j$  are monomial in the parameters  $a_k, b_k, c_k, d_k, e_k$  (or  $a_k, b_k$  if  $2d \notin \mathbb{N}$ ) with degree  $j$  and with the same parity of  $\mathfrak{L}_i^j$ .

Setting

$$\varphi = \tilde{\mathfrak{L}} - \mathfrak{L}, \quad \mathfrak{L}^2 = \sum_i P_i^2 \mathfrak{L}_i^2, \quad \mathfrak{L}^3 = \sum_i P_i^3 \mathfrak{L}_i^3, \dots,$$

we can rewrite the homomorphism condition (7.4) in the following way:

$$(5.7) \quad [\varphi(f), \mathfrak{L}_g] + [\mathfrak{L}_f, \varphi(g)] - \varphi([f, g]) + \sum_{i,j>0} [\mathfrak{L}_f^i, \mathfrak{L}_g^j] = 0,$$

or equivalently

$$(5.8) \quad \delta\varphi + \frac{1}{2}\varphi \vee \varphi = 0,$$

where  $\delta\varphi$  stands for differential of the cochain  $\varphi$  and  $\vee$  is the standard *cup-product* defined, for arbitrary linear maps  $a, b : \mathfrak{g} \longrightarrow \text{End}(V)$  with  $\mathfrak{g}$  a Lie superalgebra and  $V$  a vector superspace, by:

$$(5.9) \quad (a \vee b)(x, y) = (-1)^{xb}[a(x), b(y)] + (-1)^{a(x+b)}[b(x), a(y)],$$

so that, if  $a$  and  $b$  are even maps then

$$(a \vee b)(x, y) = [a(x), b(y)] + [b(x), a(y)].$$

From (5.8) we obtain the following equation for any  $\mathfrak{L}^k$ :

$$(5.10) \quad \delta\mathfrak{L}^k + \frac{1}{2} \sum_{i+j=k} \mathfrak{L}^i \vee \mathfrak{L}^j = 0.$$

The first non-trivial relation

$$\delta\mathfrak{L}^2 + \frac{1}{2} \mathfrak{L}^1 \vee \mathfrak{L}^1 = 0$$

gives the first obstruction to integration of an infinitesimal deformation. That is,  $\mathfrak{L}^1 \vee \mathfrak{L}^1$  must be a coboundary.

It is easy to check that for any two 1-cocycles  $C_1$  and  $C_2 \in Z^1(\mathfrak{g}, \text{End}(V))$ , the bilinear map  $C_1 \vee C_2$  is a 2-cocycle. Moreover, if one of the cocycles  $C_1$  or  $C_2$  is a coboundary, then  $C_1 \vee C_2$  is a 2-coboundary. Therefore, we naturally deduce that the operation (5.9) defines a bilinear map:

$$(5.11) \quad H^1(\mathfrak{g}, \text{End}(V)) \otimes H^1(\mathfrak{g}, \text{End}(V)) \longrightarrow H^2(\mathfrak{g}, \text{End}(V)).$$

All the obstructions lie in  $H^2(\mathfrak{g}, \text{End}(V))$  and they are in the image of  $H^1(\mathfrak{g}, \text{End}(V))$  under the cup-product. Thus, we describe in the following section the cup-product  $H^1 \vee H^1$ .

## 6. THE CUP-PRODUCT $H^1 \vee H^1$

We have to distinguish two cases:

### 6.1. Case 1: $2d \notin \mathbb{N}$ .

**Theorem 6.1.** *If  $2d \notin \mathbb{N}$  then the image  $H^1 \vee H^1$  of  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{d-\frac{k}{2}, d-\frac{k}{2}})$  under the cup-product is a 2-dimensional subspace of  $H_{\text{diff}}^2(\mathfrak{osp}(2|2), \mathfrak{D}_{d-\frac{k}{2}, d-\frac{k}{2}})$  spanned by*

$$\Omega_1 = \omega_k \vee \tilde{\omega}_k \quad \text{and} \quad \Omega_2 = \tilde{\omega}_k \vee \tilde{\omega}_k.$$

*Proof.* In this case, the space  $H^1 \vee H^1$  is generated by the three cup-products:  $\omega_k \vee \omega_k$ ,  $\omega_k \vee \tilde{\omega}_k$  and  $\tilde{\omega}_k \vee \tilde{\omega}_k$ . But it is easily check that  $\omega_k \vee \omega_k = 0$ . So, we have to prove that  $\Omega_1$  and  $\Omega_2$  are nontrivial 2-cocycles which are linearly independent. That is, the equation

$$(6.1) \quad a\Omega_1 + b\Omega_2 = \delta B,$$

where  $a, b \in \mathbb{R}$  and  $B \in C_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{d-\frac{k}{2}, d-\frac{k}{2}})$ , has a solution if and only if  $a = b = 0$ .

First of all, we have

$$\begin{aligned} \Omega_1(g, h) &= -(-1)^g \partial_2 g \bar{\eta}_1 h' - (-1)^g \theta_2 \bar{\eta}_2 \bar{\eta}_1 g \bar{\eta}_2 h' - (-1)^{gh} (g \leftrightarrow h), \\ \Omega_2(g, h) &= 2 \left[ (-1)^g (2d - k) \partial_2 g \partial_x \partial_2 h + (-1)^g \partial_2 g (\partial_2 h - \theta_1 \partial_1 \partial_2 h) \partial_x \right. \\ &\quad \left. + (-1)^{g+h} \partial_2 g \bar{\eta}_1 \partial_2 h \partial_1 + (-1)^h \theta_2 \partial_2 g \partial_x \partial_2 h \partial_2 \right] - (-1)^{gh} (g \leftrightarrow h). \end{aligned}$$

Now, for  $\alpha = (i, j, k)$ , we denote by  $\partial^\alpha = \partial_x^i \partial_1^j \partial_2^k$ . Then, by considering the equation (6.1), we can write

$$(6.2) \quad B(h) = \sum_{\alpha, \beta} A_{\alpha, \beta} \partial^\alpha (h) \partial^\beta \quad \text{where} \quad A_{\alpha, \beta} = A_{\alpha, \beta}^0 + \theta_1 A_{\alpha, \beta}^1 + \theta_2 A_{\alpha, \beta}^2 + \theta_1 \theta_2 A_{\alpha, \beta}^{12}.$$

One obtains

$$\begin{aligned} B(1) &= \sum_{\beta} A_{000, \beta} \partial^\beta \\ B(x) &= \sum_{\beta} (A_{000, \beta} x + A_{100, \beta}) \partial^\beta \\ B(x^2) &= \sum_{\beta} (A_{000, \beta} x^2 + 2x A_{100, \beta} + 2A_{200, \beta}) \partial^\beta \\ B(\theta_1) &= \sum_{\beta} (A_{000, \beta} \theta_1 + A_{010, \beta}) \partial^\beta \\ B(\theta_2) &= \sum_{\beta} (A_{000, \beta} \theta_2 + A_{001, \beta}) \partial^\beta \\ B(x\theta_1) &= \sum_{\beta} (A_{000, \beta} x\theta_1 + A_{100, \beta} \theta_1 + A_{010, \beta} x + A_{110, \beta}) \partial^\beta \\ B(x\theta_2) &= \sum_{\beta} (A_{000, \beta} x\theta_2 + A_{100, \beta} \theta_2 + A_{001, \beta} x + A_{101, \beta}) \partial^\beta \\ B(\theta_1 \theta_2) &= \sum_{\beta} (A_{000, \beta} \theta_1 \theta_2 + A_{010, \beta} \theta_2 - A_{001, \beta} \theta_1 + A_{011, \beta}) \partial^\beta \end{aligned}$$

Let us recall that

$$\begin{aligned} \delta B(g, h) : &= \mathfrak{L}_g^{\lambda, \mu} B(h) - (-1)^{hg} \mathfrak{L}_g^{\lambda, \mu} B(g) - B([g, h]) \\ &= \partial_x B(h) - \frac{1}{2} (-1)^g (\bar{\eta}_1 g \bar{\eta}_1 B(h) + \bar{\eta}_2 g \bar{\eta}_2 B(h)) \\ &\quad + \mu \partial_x g B(h) - B(h) (g \partial_x - \frac{1}{2} (-1)^g (\bar{\eta}_1 g \bar{\eta}_1 + \bar{\eta}_2 g \bar{\eta}_2) + \lambda \partial_x g) \\ &\quad - (-1)^{gh} (h \partial_x B(g) - \frac{1}{2} (-1)^h (\bar{\eta}_1 h \bar{\eta}_1 B(g) + \bar{\eta}_2 h \bar{\eta}_2 B(g)) \\ &\quad + \mu \partial_x h B(g) - B(g) (h \partial_x - \frac{1}{2} (-1)^h (\bar{\eta}_1 h \bar{\eta}_1 + \bar{\eta}_2 h \bar{\eta}_2) + \lambda \partial_x h)) \\ &\quad - B(g \partial_x h - \partial_x g h - \frac{1}{2} (-1)^g (\bar{\eta}_1 g \bar{\eta}_1 h + \bar{\eta}_2 g \bar{\eta}_2 h)) \end{aligned}$$

Now, considering the terms in  $f$  in (6.1) for  $(g, h) = (\theta_2, \theta_2)$  then for  $(g, h) = (x\theta_2, \theta_2)$ , we get

$$(6.3) \quad -\lambda A_{001, 001}^0 + \frac{1}{4} A_{101, 000}^2 = -4\lambda b.$$

Similarly, the terms in  $\theta_1 f$  for  $(g, h) = (\theta_1, \theta_1)$  then for  $(g, h) = (x\theta_1, \theta_1)$  give

$$(6.4) \quad \frac{1}{4} A_{110, 000}^1 - \lambda A_{010, 010}^0 = 0.$$

The terms in  $\theta_2 f$  for  $(g, h) = (\theta_2, \theta_1 \theta_2)$  then for  $(g, h) = (x\theta_2, \theta_1 \theta_2)$  give

$$(6.5) \quad \lambda A_{001, 001}^0 - \lambda A_{011, 001}^1 - \frac{1}{4} A_{100, 000}^0 - \frac{1}{2} A_{101, 000}^2 + \frac{1}{4} A_{110, 000}^1 = 4b\lambda.$$



Considering the terms in  $\partial_1 f$  for  $(g, h) = (\theta_1, \theta_1 \theta_2)$  and  $(g, h) = (x\theta_1, \theta_1)$  we obtain

$$(6.6) \quad -\lambda A_{011,010}^2 - \lambda A_{010,010}^0 + \frac{1}{2} A_{110,000}^1 - \frac{1}{4} A_{101,000}^2 + \frac{1}{4} A_{100,000}^0 = 0.$$

Now, we consider the terms in  $\partial_2 f$  respectively for  $(g, h) = (\theta_2, \theta_1 \theta_2)$  and for  $(g, h) = (\theta_1, \theta_1 \theta_2)$  then we obtain

$$(6.7) \quad \begin{cases} -\frac{1}{2} A_{001,001}^0 + \frac{1}{2} A_{010,010}^0 + \frac{1}{4} A_{011,010}^2 = 2b \\ -\frac{1}{2} A_{001,001}^0 + \frac{1}{4} A_{011,001}^1 + \frac{1}{2} A_{010,010}^0 = 0 \end{cases}$$

On the other hand, for  $(g, h) = (\theta_2, \theta_2)$  and for  $(g, h) = (\theta_1, \theta_1)$  we consider the terms in  $\partial_x f$  in (6.1) then we find

$$(6.9) \quad \begin{cases} \frac{1}{4} A_{000,100}^0 + \frac{1}{2} A_{001,100}^2 - \frac{3}{2} A_{001,001}^0 = -4b. \\ \frac{1}{4} A_{000,100}^0 + \frac{1}{2} A_{010,100}^2 - \frac{3}{2} A_{010,010}^0 = 0. \end{cases}$$

For  $(g, h) = (\theta_2, \theta_1 \theta_2)$  the terms in  $\theta_1 \partial_x f$  give

$$(6.11) \quad -\frac{3}{4} A_{001,100}^2 - \frac{1}{4} A_{011,100}^{12} - \frac{1}{2} A_{000,100}^0 + \frac{3}{4} A_{001,001}^0 - \frac{3}{4} A_{011,001}^1 = 2b$$

and for  $(g, h) = (\theta_1, \theta_1 \theta_2)$  the terms in  $\theta_2 \partial_x f$  imply

$$(6.12) \quad \frac{3}{4} A_{010,100}^2 + \frac{1}{4} A_{011,100}^{12} + \frac{1}{2} A_{000,100}^0 - \frac{3}{4} A_{010,010}^0 - \frac{3}{4} A_{011,010}^2 = 0$$

Now, combining equations coming from substituting (6.3) into (6.4), adding (6.5) and (6.6), (6.7) and (6.8), substituting (6.9) into (6.10) and adding (6.11) and (6.12), we immediately find  $b = 0$

To complete the proof we proceed similarly as before, therefore we get

$$\begin{cases} \frac{1}{4} A_{101,000}^1 + \frac{1}{4} A_{110,000}^2 = a \\ \frac{1}{4} A_{101,000}^1 + \frac{1}{2} A_{110,000}^2 = a \\ \frac{1}{2} A_{101,000}^1 + \frac{1}{4} A_{110,000}^2 = a. \end{cases}$$

Thus, it is easy to see that  $a = 0$ . So we obtain the claim.  $\square$

## 6.2. Case 2: $2d = m \in \mathbb{N}$ .

**Theorem 6.2.** *If  $2d = m \in \mathbb{N}$  then the image  $H^1 \vee H^1$  of  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{d-\frac{k}{2}, d-\frac{k}{2}})$  under the cup-product is a 6-dimensional subspace of  $H_{\text{diff}}^2(\mathfrak{osp}(2|2), \mathfrak{D}_{-\frac{k}{2}, \frac{k}{2}})$  spanned by*

$$\Phi_1 = \gamma_k \vee \tilde{\Gamma}_k, \quad \Phi_2 = \gamma_k \vee \bar{\Gamma}_k, \quad \Phi_3 = \tilde{\gamma}_k \vee \tilde{\Gamma}_k, \quad \Phi_4 = \tilde{\gamma}_k \vee \bar{\Gamma}_k, \quad \Phi_5 = \tilde{\Gamma}_k \vee \gamma_{-k} \text{ and } \Phi_6 = \bar{\Gamma}_k \vee \tilde{\gamma}_{-k}.$$

Proof. In this case, the space  $H^1 \vee H^1$  is generated by the following twelve cup-products:  
 $\Phi_1 = \gamma_k \vee \tilde{\Gamma}_k, \quad \Phi_2 = \gamma_k \vee \bar{\Gamma}_k, \quad \Phi_3 = \tilde{\gamma}_k \vee \tilde{\Gamma}_k, \quad \Phi_4 = \tilde{\gamma}_k \vee \bar{\Gamma}_k, \quad \Phi_5 = \tilde{\Gamma}_k \vee \gamma_{-k}, \quad \Phi_6 = \bar{\Gamma}_k \vee \tilde{\gamma}_{-k},$   
 $\Phi_7 = \gamma_k \vee \Gamma_k, \quad \Phi_8 = \Gamma_k \vee \tilde{\gamma}_{-k}, \quad \Phi_9 = \tilde{\Gamma}_k \vee \tilde{\gamma}_{-k}, \quad \Phi_{10} = \Gamma_k \vee \gamma_{-k}, \quad \Phi_{11} = \bar{\Gamma}_k \vee \gamma_{-k} \text{ and } \Phi_{12} = \tilde{\gamma}_k \vee \Gamma_k.$

By a straightforward computation, we check that

$$\Phi_7 = 0, \quad \Phi_8 = -\Phi_1, \quad \Phi_{10} = \Phi_{11} = -\Phi_2, \quad \Phi_9 = -\Phi_3, \quad \text{and} \quad \Phi_{12} = \Phi_4 + \Phi_5$$

where

$$\begin{aligned}\Phi_1(g, h) &= (-1)^k \left[ (-1)^h g' \partial_2 h \theta_2 - (k+1) g' \bar{\eta}_1 \partial_2 h \theta_1 \theta_2 \right] \partial_{\mathbf{x}}^{k+1} + (-1)^k [k g' \bar{\eta}_1 \partial_2 h \theta_1 - (-1)^h g' \partial_2 H] \partial_{\mathbf{x}}^k \partial_2 \\ &\quad - (-1)^k (k+1) g' \bar{\eta}_1 \partial_2 h \theta_2 \partial_{\mathbf{x}}^k \partial_1 - (-1)^k k g' \bar{\eta}_1 \partial_2 H \partial_{\mathbf{x}}^{k-1} \partial_1 \partial_2 - (-1)^{gh} (g \leftrightarrow h)\end{aligned}$$

$$\begin{aligned}\Phi_2(g, h) &= (-1)^k \left[ (k-1) g' h'' \theta_2 - (-1)^h g' \bar{\eta}_2 h' \right] \partial_{\mathbf{x}}^{k-1} \partial_1 - (-1)^k \left[ (k-1) g' h'' \theta_1 - (-1)^h g' \bar{\eta}_1 h' \right] \partial_{\mathbf{x}}^{k-1} \partial_2 \\ &\quad + (-1)^k \left[ (k-1) g' h'' \theta_1 \theta_2 + (-1)^h g' \bar{\eta}_2 h' \theta_1 - (-1)^h g' \bar{\eta}_1 h' \theta_2 \right] \partial_{\mathbf{x}}^k \\ &\quad + (-1)^k (k-1) g' h'' \partial_{\mathbf{x}}^{k-2} \partial_1 \partial_2 - (-1)^{gh} (g \leftrightarrow h)\end{aligned}$$

$$\begin{aligned}\Phi_3(g, h) &= (-1)^k \left[ (-1)^{g+h} \partial_2 g \bar{\eta}_1 \partial_2 h + (-1)^g k \partial_2 g \partial_x \partial_2 h \theta_1 \right] \partial_{\mathbf{x}}^k \partial_2 - (-1)^{g+k} (k+1) \partial_2 g \partial_x \partial_2 h \theta_2 \partial_{\mathbf{x}}^k \partial_1 \\ &\quad - (-1)^k \left[ (-1)^{g+h} \partial_2 g \partial_1 \partial_2 h \theta_2 + (-1)^g (k+2) \partial_2 g \partial_x \partial_2 h \theta_1 \theta_2 \right] \partial_{\mathbf{x}}^{k+1} - (-1)^{k+g} \partial_2 g \partial_2 h \times \\ &\quad \left[ \partial_1 \partial_2 + \theta_2 \partial_{\mathbf{x}} \partial_1 - \theta_1 \partial_{\mathbf{x}}^{k+1} \partial_2 + \theta_1 \theta_2 \partial_{\mathbf{x}}^2 \right] - (-1)^{k+g} k \partial_2 g \partial_x \partial_2 h \partial_{\mathbf{x}}^{k-1} \partial_1 \partial_2 - (-1)^{gh} (g \leftrightarrow h)\end{aligned}$$

$$\begin{aligned}\Phi_4(g, h) &= (-1)^k \left[ (-1)^g \partial_2 g \bar{\eta}_1 h' \right] \left[ \theta_2 \partial_{\mathbf{x}}^k \partial_1 + \theta_1 \theta_2 \partial_{\mathbf{x}}^{k+1} \right] - (-1)^k \left[ k(k-1) \bar{\eta}_1 \partial_2 g h'' \theta_1 - (-1)^h k \bar{\eta}_1 \partial_2 g \bar{\eta}_1 h' \right. \\ &\quad \left. - (-1)^{g+h} k \partial_2 g h'' \right] \partial_{\mathbf{x}}^{k-1} \partial_2 + (-1)^k \left[ k^2 \bar{\eta}_1 \partial_2 g h'' \theta_2 - (-1)^h k \bar{\eta}_1 \partial_2 g \bar{\eta}_2 h' \right] \partial_{\mathbf{x}}^{k-1} \partial_1 \\ &\quad + (-1)^k \left[ k^2 \bar{\eta}_1 \partial_2 g h'' \theta_1 \theta_2 + (-1)^h k \bar{\eta}_1 \partial_2 g \bar{\eta}_2 h' \theta_1 - (-1)^h (k+1) \bar{\eta}_1 \partial_2 g \bar{\eta}_1 h' \theta_2 \right. \\ &\quad \left. - (-1)^{g+h} k \partial_2 g h'' \theta_2 + (-1)^g \partial_2 g \bar{\eta}_2 h' \right] \partial_{\mathbf{x}}^k + (-1)^k k(k-1) \bar{\eta}_1 \partial_2 g h'' \partial_{\mathbf{x}}^{k-2} \partial_1 \partial_2 \\ &\quad + (-1)^k \left[ (-1)^g \partial_2 (g) \bar{\eta}_1 h' + (-1)^g \theta_2 \bar{\eta}_2 \bar{\eta}_1 g \bar{\eta}_2 h' \right] \left[ \partial_{\mathbf{x}}^{k-1} \partial_1 \partial_2 - \theta_1 \partial_{\mathbf{x}}^k \partial_2 \right] - (-1)^{gh} (g \leftrightarrow h).\end{aligned}$$

$$\begin{aligned}\Phi_5(g, h) &= (-1)^k \left[ k(k-1) \bar{\eta}_1 \partial_2 g h'' \theta_1 - (-1)^h k \bar{\eta}_1 \partial_2 g \bar{\eta}_1 g' - (-1)^{g+h} k \partial_2 g h'' \right] \partial_{\mathbf{x}}^{k-1} \partial_2 - (-1)^k \left[ k^2 \bar{\eta}_1 \partial_2 g h'' \theta_2 \right. \\ &\quad \left. - (-1)^h k \bar{\eta}_1 (\partial_2 (g) \bar{\eta}_2 h' \right] \partial_{\mathbf{x}}^{k-1} \partial_1 - (-1)^k k(k-1) \bar{\eta}_1 \partial_2 g h'' \partial_{\mathbf{x}}^{k-2} \partial_1 \partial_2 - (-1)^k k \bar{\eta}_1 \partial_2 g h' \partial_{\mathbf{x}}^{k-1} \partial_1 \partial_2 \\ &\quad - (-1)^k \left[ k \bar{\eta}_1 \partial_2 g h' \theta_2 - (-1)^{g+h} \bar{\eta}_1 \theta_2 \partial_2 g h' \right] \partial_{\mathbf{x}}^k \partial_1 + (-1)^k \left[ k \bar{\eta}_1 \partial_2 g h' \theta_1 - (-1)^{h+g} \partial_2 g h' \right] \partial_{\mathbf{x}}^k \partial_2 \\ &\quad - (-1)^k \left[ k \bar{\eta}_1 \partial_2 g h' \theta_1 \theta_2 - (-1)^{g+h} \partial_2 (g) h' \theta_2 + (-1)^{g+h} \bar{\eta}_1 (\theta_2 \partial_2 g) h' \theta_1 \right] \partial_{\mathbf{x}}^{k+1} \\ &\quad + (-1)^k \left[ (-1)^h (k+1) \bar{\eta}_1 \partial_2 g \bar{\eta}_1 g' \theta_2 - k^2 \bar{\eta}_1 \partial_2 g h'' \theta_1 \theta_2 + (-1)^{g+h} k \partial_2 g h'' \theta_2 - (-1)^h k \bar{\eta}_1 \partial_2 g \bar{\eta}_2 h' \theta_1 \right. \\ &\quad \left. - (-1)^g \partial_2 g \bar{\eta}_2 h' \right] \partial_{\mathbf{x}}^k - (-1)^{gh} (g \leftrightarrow h)\end{aligned}$$

$$\begin{aligned}\Phi_6(g, h) &= -(-1)^k \left[ (-1)^g \bar{\eta}_2 g' \theta_2 \bar{\eta}_2 \bar{\eta}_1 h + (-1)^g \bar{\eta}_1 g' \partial_2 h \right] \left[ \partial_{\mathbf{x}}^{k-1} \partial_1 \partial_2 + \theta_2 \partial_{\mathbf{x}}^k \partial_1 - \theta_1 \partial_{\mathbf{x}}^k \partial_2 + \theta_1 \theta_2 \partial_{\mathbf{x}}^{k+1} \right] \\ &\quad - (-1)^k \left[ k(k-1) g'' \bar{\eta}_1 \partial_2 h + (-1)^g (k-1) \bar{\eta}_1 g' \partial_x \partial_2 h \right] \partial_{\mathbf{x}}^{k-2} \partial_1 \partial_2 - \left[ (k-1)(k+1) g'' \bar{\eta}_1 \partial_2 h \theta_2 \right. \\ &\quad \left. - (-1)^{g+h} k \bar{\eta}_1 g' \theta_2 \partial_x \partial_2 h - (-1)^{g+h} (k+1) \bar{\eta}_2 g' \bar{\eta}_1 \partial_2 h \right] (-1)^k \partial_{\mathbf{x}}^{k-1} \partial_1 + \left[ k(k-1) g'' \bar{\eta}_1 \partial_2 h \theta_1 \right. \\ &\quad \left. + (-1)^g (k-1) \bar{\eta}_1 g' \partial_x \partial_2 h \theta_1 - (-1)^h (k-1) g'' \partial_2 h - (-1)^{g+h} \bar{\eta}_2 g' \theta_2 \partial_2 h \right. \\ &\quad \left. - (-1)^{g+h} (k-1) \bar{\eta}_1 g' \bar{\eta}_1 \partial_2 h \right] (-1)^k \partial_{\mathbf{x}}^{k-1} \partial_2 - (-1)^k \left[ (k+1)(k-1) g'' \bar{\eta}_1 \partial_2 (h) \theta_1 \theta_2 \right. \\ &\quad \left. - (-1)^{g+h} k \bar{\eta}_1 g' \bar{\eta}_1 \partial_2 h \theta_2 + (-1)^g k \bar{\eta}_1 g' \partial_2 \partial_x h \theta_1 \theta_2 - (-1)^h (k-1) g'' \partial_2 h \theta_2 \right. \\ &\quad \left. + (-1)^{g+h} (k+1) \bar{\eta}_2 g' \partial_1 \partial_2 h \theta_1 - (-1)^g \bar{\eta}_2 g' \partial_2 h \right] \partial_{\mathbf{x}}^k + (-1)^{h+k} (k-1) g'' \partial_x \partial_2 h \partial_{\mathbf{x}}^{k-2} \partial_2 \\ &\quad + (-1)^k \left[ -(-1)^h (k-1) g'' \partial_x \partial_2 h \theta_2 - (-1)^g \bar{\eta}_2 g' \partial_x \partial_2 h \right] \partial_{\mathbf{x}}^{k-1} - (-1)^{gh} (g \leftrightarrow h)\end{aligned}$$

The proof is almost identical to the previous theorem. Indeed, we have to prove that  $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5$ , and  $\Phi_6$  are nontrivial 2-cocycles which are linearly independent. That is, the equation

$$(6.13) \quad a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 + a_4\Phi_4 + a_5\Phi_5 + a_6\Phi_6 = \delta B$$

has a solution if and only if  $a_1 = \dots = a_6 = 0$ . Of course we can express  $B$  as in (6.2).

Here, we just give the parameters of identifications which allow us to obtain the result. For (6.13), considering the terms in  $\partial_x^k \partial_2$  for  $(g, h) = (1, \theta_2)$  and  $(g, h) = (x, \theta_2)$  and the terms in  $\partial_x^k \partial_1$  for  $(g, h) = (1, \theta_1)$  and  $(g, h) = (x, \theta_1)$ , we get  $A_{100, k01}^2 = -4(-1)^k(a_1 - a_5)$ . But the terms in  $\theta_1 \partial_x^k \partial_2$ , for  $(g, h) = (1, \theta_1 \theta_2)$  and then for  $(g, h) = (x, \theta_1 \theta_2)$  give  $A_{100, k01}^2 = (-1)^k 2(k-1)(a_1 - a_5)$ , therefore  $a_1 = a_5$ .

Similarly, considering the terms in  $\theta_1 \theta_2 \partial_x^k$ , for  $(g, h) = (1, x), (1, x^2), (x, x^2)$  we obtain  $a_2 = 0$ .

Now, since  $a_1 = a_5$  and  $a_2 = 0$ , the terms in  $\theta_1 \theta_2 \partial_x^k$ , for  $(g, h) = (1, \theta_1 \theta_2), (x, \theta_1 \theta_2), (x^2, \theta_1 \theta_2)$  give  $a_4 - a_6 + a_5 = 0$ . Thus, the terms in  $\partial_x^{k-2} \partial_1 \partial_2$ , for  $(g, h) = (x\theta_1, x\theta_2), (\theta_1, \theta_2), (x\theta_1, \theta_2), (x\theta_2, \theta_1)$  give  $A_{101, (k-1)01}^0 - A_{110, (k-1)10}^0 = (-1)^k 4a_6$ . On the other hand, the terms in  $\partial_x^{k-1} \partial_2$  for  $(g, h) = (\theta_1, \theta_1 \theta_2), (x\theta_1, \theta_1 \theta_2)$  and the terms in  $\partial_x^{k-1} \partial_1$  for  $(g, h) = (\theta_2, \theta_1 \theta_2), (x\theta_2, \theta_1 \theta_2)$ , give  $A_{101, (k-1)01}^0 - A_{110, (k-1)10}^0 = -(-1)^k \frac{4}{3} a_6$ , then  $a_6 = 0$ . So, considering the terms in  $\partial_x^k \partial_1 \partial_2$ , for  $(g, h) = (\theta_1, \theta_2)$  combined with those in  $\theta_1 \partial_x^{k-1} \partial_1 \partial_2$ , for  $(g, h) = (x^2, \theta_2), (x^2, \theta_1)$  and by those in  $\theta_1 \partial_x^{k-1} \partial_1 \partial_2$ , for  $(g, h) = (x\theta_1, \theta_2), (x\theta_2, \theta_1)$ , we get

$$A_{101, (k-1)11}^1 + A_{110, (k-1)11}^2 = -(-1)^k 2a_4 + k(A_{010, k10}^0 - A_{001, k01}^0)$$

and

$$A_{101, (k-1)11}^1 + A_{110, (k-1)11}^2 = -(-1)^k 4a_4 + k(A_{010, k10}^0 - A_{001, k01}^0).$$

Then  $a_4 = 0$  and consequently  $a_1 = a_5 = 0$ .

Finally, we consider the terms in  $\partial_x^k \partial_2$ , for  $(g, h) = (\theta_2, \theta_1 \theta_2)$ , the terms in  $\partial_x^k \partial_1$ , for  $(g, h) = (\theta_1, \theta_1 \theta_2)$ , the terms in  $\partial_x^{k-1} \partial_1 \partial_2$ , for  $(g, h) = (x\theta_2, \theta_1 \theta_2), (x\theta_1, \theta_1 \theta_2), (x\theta_2, \theta_2), (x\theta_1, \theta_1)$  and the terms in  $\partial_x^k \partial_1 \partial_2$ , for  $(g, h) = (\theta_2, \theta_1 \theta_2), (\theta_1, \theta_1 \theta_2), (\theta_2, \theta_2), (\theta_1, \theta_1)$ , then we have

$$A_{001, k10}^0 + A_{010, k01}^0 = -(-1)^k \frac{2}{3} a_3$$

and

$$A_{010, k01}^0 + A_{001, k10}^0 = -(-1)^k a_3.$$

Then  $a_3 = 0$ . □

We formulate a conjecture on the structure of the second cohomology space.

*Conjecture 6.1.* One has  $H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}) \vee H_{\text{diff}}^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}) = H_{\text{diff}}^2(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu})$ .

This conjecture is an important open problem concerning the computation of second cohomology spaces which are generally difficult to derive. It turns out that a positive confirmation of this type of conjecture is a crucial result as obtained by Arnal, Ben Ammar and Dali [4] where they proved that  $H^2(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}) = H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu}) \vee H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda, \mu})$ .

## 7. INTEGRABILITY CONDITIONS

Now, we consider an infinitesimal deformation

$$(7.1) \quad \tilde{\mathfrak{L}} = \mathfrak{L} + \mathfrak{L}^1$$

of the natural action  $\mathfrak{L}$  of  $\mathfrak{osp}(2|2)$  on the space  $\mathfrak{S}_d^2$  and we study the necessary and sufficient conditions to extend it to a formal one:

$$(7.2) \quad \tilde{\mathfrak{L}} = \mathfrak{L} + \mathfrak{L}^1 + \sum_i P_i^2 \mathfrak{L}_i^2 + \sum_i P_i^3 \mathfrak{L}_i^3 + \dots$$

where

$$\mathfrak{L}^1 = \begin{cases} \sum_{k \geq 0} (a_k \omega_k + b_k \tilde{\omega}_k) & \text{if } 2d \notin \mathbb{N} \\ \sum_{k \leq m} (a_k \gamma_k + b_k \tilde{\gamma}_k) + \sum_{k=1}^m (c_k \Gamma_k + d_k \tilde{\Gamma}_k + e_k \bar{\Gamma}_k) & \text{if } 2d = m \in \mathbb{N} \end{cases}$$

and the higher order terms  $\mathfrak{L}_i^2, \mathfrak{L}_i^3, \dots$  are linear maps from  $\mathfrak{osp}(2|2)$  to  $\text{End}(\mathfrak{S}_d^2)$  such that the map

$$(7.3) \quad \tilde{\mathfrak{L}} : \mathfrak{osp}(2|2) \rightarrow \mathbb{C}[[a_k, b_k, c_k, d_k, e_k]] \otimes \text{End}(\mathfrak{S}_d^2),$$

satisfies the homomorphism condition:

$$(7.4) \quad \tilde{\mathfrak{L}}_{[f,g]} = [\tilde{\mathfrak{L}}_f, \tilde{\mathfrak{L}}_g],$$

and  $P_i^j$  are monomial in the independent parameters  $a_k, b_k, c_k, d_k, e_k$  (or  $a_k, b_k$  if  $2d \notin \mathbb{N}$ ) with degree  $j$  and with the same parity of  $\mathfrak{L}_i^j$ .

The following theorems are our main results. We have to distinguish two cases.

**7.1. Case 1:**  $2d \notin \mathbb{N}$ . In this case, we have  $\mathfrak{L}^1 = \sum_{k \geq 0} (a_k \omega_k + b_k \tilde{\omega}_k)$ .

**Theorem 7.1.** *The following conditions are necessary and sufficient for integrability of the infinitesimal deformation (5.3)*

$$(7.5) \quad b_k = 0, \quad \text{for all } k \geq 0.$$

Moreover, any formal deformation is equivalent to its infinitesimal part which is of the form:

$$(7.6) \quad \mathfrak{L} + \sum_{k \geq 0} a_k \omega_k.$$

That is, formal deformations are classified by the subspace of  $H^1(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda, \mu}^2)$  spanned by the cohomological classes of the 1-cocycles  $\omega_k$ .

Proof. The condition (7.4) gives, for the second-order terms, the following equation

$$(7.7) \quad \delta \mathfrak{L}^2 = \frac{1}{2} \sum_{k \geq 0} (a_k b_k \Omega_1 + b_k^2 \Omega_2).$$

Thus, the right hand side of (7.7) must be a coboundary. But, by Theorem 6.1,  $\Omega_1$  and  $\Omega_2$  are linearly independent nontrivial 2-cocycles, therefore  $a_k b_k = b_k^2 = 0$  for all  $k \geq 0$ . Thus, the conditions (7.5) are necessary.

Now, we show that these conditions are sufficient. The solutions  $\mathfrak{L}^k$  of the Maurer-Cartan equations (5.10) are defined up to a 1-cocycle and it has been shown in works [2] and [11] that different choices of solutions correspond to equivalent deformations. Thus, we can always reduce  $\mathfrak{L}^k$ , for  $k = 2$  to zero by equivalence. Then, by recurrence, the highest-order terms  $\mathfrak{L}^k$  with  $k \geq 3$ , also satisfy the equation  $\delta(\mathfrak{L}^k)$  and can also be reduced to the identically zero map. One obviously obtains a deformation (which is of order 1 in  $a_k$ ).  $\square$

**7.2. Case 2:**  $2d = m \in \mathbb{N}$ . In this case we have  $\mathfrak{L}^1 = \sum_{k \leq m} (a_k \gamma_k + b_k \tilde{\gamma}_k) + \sum_{k=1}^m (c_k \Gamma_k + d_k \tilde{\Gamma}_k + e_k \bar{\Gamma}_k)$ .

**Theorem 7.2.** *The following conditions are necessary and sufficient for integrability of the infinitesimal deformation (5.3): For all  $k \geq 0$ ,*

$$\begin{cases} a_k d_k - c_k b_{-k} = 0 \\ a_k e_k - c_k a_{-k} - e_k a_{-k} = 0 \\ b_k d_k - d_k b_{-k} = 0 \\ b_k c_k = -b_k e_k = -d_k a_{-k} \\ e_k b_{-k} = 0. \end{cases}$$

Moreover, any formal deformation is equivalent to its infinitesimal part.

Thus, similarly to the first case, these above conditions give a classification of the formal deformations.

Proof. In this case, the equation (7.4) can be expressed as follows

$$\begin{aligned} \delta \mathfrak{L}^2 &= \frac{1}{2} \sum_{k \geq 0} [(a_k d_k - c_k b_{-k}) \Phi_1 + (a_k e_k - c_k a_{-k} - e_k a_{-k}) \Phi_2 + (b_k d_k - d_k b_{-k}) \Phi_3 \\ (7.8) \quad &+ (b_k c_k + b_k e_k) \Phi_4 + (b_k c_k + d_k a_{-k}) \Phi_5 + e_k b_{-k} \Phi_6] \end{aligned}$$

The second order integrability conditions are determined by the fact that the map 2-cocycles  $\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6$  are non-trivial, which is proved in Theorem 6.2. As above, these conditions are sufficient and the terms  $\mathfrak{L}^k$  with  $k \geq 2$  can be chosen identically zero.

**Example 7.3.** Let us consider  $d = \frac{m}{2} \in \mathbb{N}$ , let  $b_k = d_k = 0$ ;  $k \in \mathbb{Z}$  and  $c_k = e_k$ . So, we obtain the following deformation of  $\mathfrak{S}_d^2$  with two family of independent parameters

$$\tilde{\mathfrak{L}} = \mathfrak{L} + \sum_{k \leq m} 2a_{-k} \gamma_k + \sum_{k=1}^m c_k (\Gamma_k + \bar{\Gamma}_k).$$

Of course it is easy to give many other examples of true deformations.

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